

Stability of Recursive Digital Filters

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If $B(z)$ is an absolutely convergent power series on the unit disk, the requirements for the reciprocal of B to also be absolutely convergent on the unit disc are well known. These requirements are that B does not vanish on the unit disc. This paper gives an alternate characterization of the corresponding theorem in two-variables along with a new proof.

1. INTRODUCTION

Digital filtering is used to process discrete data. It has been applied in many environments, some of which require the processing of biomedical data, seismic data, and various other "digital signals" which have been sampled from continuous data.

One of the types of digital filters is the "recursive filter." Such filters may be used to process two-dimensional data, such as digitized photographic images and seismic data sections. One of the intrinsic problems associated with recursive filters is that of stability.

DEFINITION 1. A two-dimensional function with z -transform

$$H(z_1, z_2) = \sum_{j=0}^J \sum_{k=0}^K h_{j,k} z_1^j z_2^k, \quad 0 \leq j, \quad k \leq \infty \quad (1)$$

is called stable if and only if the power series (1) converges absolutely for $|z_1| \leq 1, |z_2| \leq 1$.

The definition has physical meaning, in the sense that it ensures the amplitude sum of the response of any system is finite.

DEFINITION 2. A two-dimensional function with z -transform $B(z_1, z_2)$ is called recursively stable if and only if the function $A(z_1, z_2) = 1/B(z_1, z_2)$ can be written as a power series, which is stable.

The generally theorem has appeared in reference [1], and is given by

THEOREM 1. *Given that $B(z_1, z_2)$ is a power series in (z_1, z_2) , for the coefficients of the expansion of $1/B(z_1, z_2)$ in positive powers of z_1 and z_2 to converge absolutely, it is necessary and sufficient that $B(z_1, z_2)$ not be zero for $|z_1|$ and $|z_2|$ simultaneously less than or equal to 1.*

The stability problem, as it relates to the design at two-dimensional digital filters, has been the subject of several investigations [1, 2]. Huang's simplification [2] of the stability problem is essentially as follows; $B(z_1, z_2) \neq 0$ for $|z_1| \leq 1$ and $|z_2| \leq 1$ if and only if the following two conditions hold:

$$B(z_1, 0) \neq 0, \quad |z_1| \leq 1, \quad (2)$$

$$B(z_1, z_2) \neq 0, \quad |z_1| = 1, \quad |z_2| \leq 1. \quad (3)$$

Let us briefly indicate how these constraints may be checked. First, the checking of (2) is straightforward since $B(z_1, 0)$ is a single variable polynomial, and there are a number of Hurwitz tests for determining whether or not all its roots lie outside the unit circle.

Unfortunately, condition (3) must be checked by a tedious finite procedure, based on a bilinear transformation in both variables. We shall omit further discussion of testing of stability, the intent herein being to obtain an alternate characterization of the stability test.

2. STABILITY THEOREM. *A function with z -transform $B(z_1, z_2)$ is recursively stable if and only if there exists $\epsilon_1 > 0$ such that $B(z_1, z_2) \neq 0$ and regular for $|z_1| \leq 1 + \epsilon_1$ and $|z_2| \leq 1 + \epsilon_1$ simultaneously.*

Proof. Suppose $B(z_1, z_2)$ is recursively stable, we then know from Theorem 1 that $B(z_1, z_2) \neq 0$ for $|z_1| \leq 1$ and $|z_2| \leq 1$ simultaneously. Let $z(z_2^*)$ and $|z(z_2^*)|$ be given by

$$z(z_2^*) = \{z_1: B(z_1, z_2^*) = 0 \text{ and } |z_2^*| \leq 1\},$$

$$|z(z_2^*)| = \{|z_1|: z_1 \text{ is in } z(z_2^*)\}.$$

Now $|z(z_2^*)|$ is obviously bounded below. Thus we may let $L_1 = \inf |z(z_2^*)|$.

Assume $L_1 \leq 1$. We may choose x_j in $z(z_2^*)$ such that $|x_{j+1}| \leq |x_j|$ for all j and such that $\lim |x_j| = L_1$. Since x_j is in the compact set $\{x: |x| \leq |x_1|\}$, then $\{x_j\}$ has a convergent subsequence $\{y_j\}$. If $y = \lim y_j$, then $|y| \leq 1$. Let z_2^* be such that $|z_2^*| \leq 1$. We then have $B(y, z_2^*) = \lim B(y_j, z_2^*) = 0$. However, B can not vanish at (y, z_2^*) , as $|y| \leq 1$ and $|z_2^*| \leq 1$. Therefore, we must conclude that $L_1 > 1$. Similarly, we see that $L_2 = \inf |z(z_1^*)| > 1$.

Let us choose L as follows $L = \min(L_1, L_2)$. It is easily seen that $B(z_1, z_2) \neq 0$ for

$$|z_j| \leq (1 + L)/2$$

for $j = 1, 2$. For ϵ_1 given by $\epsilon_1 = (L - 1)/2$, we know $B(z_1, z_2) \neq 0$ for $|z_j| \leq 1 + \epsilon_1$, $j = 1, 2$. We then have $1/B(z_1, z_2)$ regular within the closure of the preceding bicylinder. We immediately see that $1/B(z_1, z_2)$ can be written as the power series

$$\sum_{j,k=0}^{\infty} \alpha_{j,k} z_1^j z_2^k,$$

where $\{a_{j,k}\}$ is given by the Cauchy integral formula:

$$a_{j,k} = \frac{1}{(2\pi i)^2} \int_{|z_1|=1+\epsilon_1} dz_1 \int_{|z_2|=1+\epsilon_1} \frac{dz_2}{B(z_1, z_2) z_1^{j+1} z_2^{k+1}}.$$

This series is absolutely convergent for $|z_j| < 1 + \epsilon_1$, $j = 1, 2$.

If we assume that $B(z_1, z_2) \neq 0$ and regular for $|z_1| \leq 1 + \epsilon_1$ and $|z_2| \leq 1 + \epsilon_2$ for any fixed $\epsilon_1 > 0$, an appeal to Theorem 1 yields the desired result.

REFERENCES

1. C. FARMER AND B. BEDNAR, Stability of spatial digital filters, *Math. Biosci.* 14 (1972), 113-119.
2. T. S. HUANG, Stability of two-dimensional recursive filters, *IEEE Trans. Audio Electroacoust.* AU-20 (1972), 158-163.